

A TALE OF THREE HOMOTOPIES

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ABSTRACT. For a Koszul operad \mathcal{P} , there are several existing approaches to the notion of a homotopy between strong homotopy morphisms of strong homotopy \mathcal{P} -algebras. Some of those approaches are known to give rise to the same notions. We exhibit the missing links between those notions, thus putting them all into the same framework. The main nontrivial ingredient in establishing this relationship is the homotopy transfer theorem for homotopy cooperads.

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INTRODUCTION

Starting from Quillen’s work on rational homotopy theory [30], equivalences between various homotopy categories of algebras have proved to be one of the key tools of homotopical algebra (for one very informative review of how this story progressed over years, see [19]). The types of algebras for which the corresponding homotopy categories have attracted most attention over years are, eloquently described by Jean-Louis Loday, “the three graces”, that is associative algebras, associative commutative algebras, and Lie algebras. However, the corresponding questions make sense for any type of algebras, or, in a more modern language, for algebras over any operad. For instance, for the algebra of dual numbers $\mathbb{k}[\epsilon]/(\epsilon^2)$ viewed as an operad with unary operations only, algebras are chain complexes, and a good understanding of the corresponding homotopy category naturally leads to the notion of a spectral sequence [22]. In general, a “nice” homotopy theory of algebras over an operad \mathcal{P} is available in the case of any Koszul operad [35]. More precisely, there are several equivalent ways to relax a notion of a dg (standing for differential graded) \mathcal{P} -algebra up to homotopy, and define appropriate homotopy morphisms of homotopy algebras.

However consistent with one another various notions of homotopy \mathcal{P} -algebras and homotopy morphisms between them are, one enters a grey area when trying to encode homotopy relations between homotopy morphisms (motivated, for instance, by the informal relationship between the categorification of \mathcal{P} -algebras and relaxing \mathcal{P} -algebras up to homotopy, see, e. g. [2, 20]). Basically, there are at least the following natural candidates to encode homotopies between morphisms:

- the *concordance* relation between homotopies, based on two different augmentations of the dg algebra $\Omega([0, 1])$ of differential forms on the interval (this notion is discussed in [31] in detail; it seems to have first appeared in unpublished work of Stasheff and Schlessinger [33] and is inspired by a paper of Bousfield and Gugenheim [4])
- several notions of homotopy relations based on the interpretation of homotopy morphisms as Maurer–Cartan elements in a certain L_∞ -algebra:

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- the *Quillen homotopy* notion (close to the above notion of concordance) suggesting that two Maurer–Cartan elements in an algebra L are homotopic if they are images of the same Maurer–Cartan element in $L[t, dt]$ under two different morphisms to L
- the *gauge homotopy* notion suggesting that the component L_0 of an L_∞ -algebra L acts on Maurer–Cartan elements, and homotopy classes are precisely orbits of that action (this also seems to originate in [33])
- the *cylinder homotopy* notion coming from the cylinder construction of the dg Lie algebra controlling Maurer–Cartan elements; such a cylinder is shown [6] to be given by the Lawrence–Sullivan construction [23]
- the notion of *operadic homotopy* suggesting that the datum of two homotopy algebras, two homotopy morphisms between them, and a homotopy between those two morphisms is the same as the datum of an algebra over a certain cofibrant replacement of the coloured operad describing the diagram

$$\begin{array}{ccc} & p & \\ X & \xrightarrow{\quad} & Y \\ & q & \end{array} \quad \parallel$$

of \mathcal{P} -algebras (this approach was pursued by Markl in [27], following the description of homotopy algebras and homotopy morphisms via algebras over minimal models of appropriate operads [26]).

The goal of this paper is to exhibit, for a Koszul operad \mathcal{P} , interrelationships between these definitions, putting the above approaches in a common context. For different notions of homotopies between Maurer–Cartan elements, it is done in a recent preprint [5]. The interplay between concordance, Quillen homotopy, and operadic homotopy is explained in this paper. In a sense, [5] and this paper share an important cornerstone: using homotopy transfer techniques to replace the dg algebra of differential forms on the interval (respectively, its dual coalgebra) by the Čech cochain (respectively, chain) complex of the interval in various constructions involved. The first such computation goes back to Cheng and Getzler [8]. The difference, however, is that while for the purposes of [5, 6] it is enough to perform a computation dual to the one of [8] and apply homotopy transfer for homotopy coassociative coalgebras (A_∞ -coalgebras) [21, 24], for the purpose of this paper we need to use homotopy transfer for homotopy cooperads [12].

The paper is organised as follows. In Section 1, we briefly recall all necessary definitions and facts of operadic homotopical algebra. In Section 2, we provide background information on the existing notions of homotopies; even though the three different notions of a homotopy between Maurer–Cartan elements in L_∞ -algebras are known to be equivalent, we spell out the corresponding definitions for the sake of completeness. In Section 3, we explain the relationship between the notion of a concordance homotopy and that of a Quillen homotopy. In Section 4, we explain the relationship between the notion of a concordance homotopy and that of an operadic homotopy. We conclude with an outline of some future directions in Section 5.

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1. OPERADIC HOMOTOPICAL ALGEBRA

We refer the reader to [24] for details on all constructions of homotopical algebra for operads, and only recall briefly the notions of immediate importance.

Notational conventions. All vector spaces are defined over a field \mathbb{k} of characteristic 0. All operads are assumed symmetric; in addition, the word operad, depending on the context, may assume the additional characteristics “coloured” and “differential graded”. To handle suspensions, we introduce a formal symbol s of degree 1. For a graded vector space L , its suspension sL is nothing but $\mathbb{k}s \otimes L$. For an augmented (co)operad \mathcal{P} (for example, for every (co)operad \mathcal{P} with $\dim \mathcal{P}(1) = 1$), we denote by $\overline{\mathcal{P}}$ its augmentation (co)ideal.

1.1. Operadic Koszul duality and homotopy (co)algebras. To an \mathbb{S} -module \mathcal{V} and an \mathbb{S} -submodule $\mathcal{R} \subset \mathcal{F}(\mathcal{V})^{(2)}$ one can associate an operad $\mathcal{P} = \mathcal{P}(\mathcal{V}, \mathcal{R})$, the universal quotient operad \mathcal{O} of $\mathcal{F}(\mathcal{V})$ for which the composite

$$\mathcal{R} \hookrightarrow \mathcal{F}(\mathcal{V}) \twoheadrightarrow \mathcal{O}$$

is zero. Similarly, to an \mathbb{S} -module \mathcal{V} and an \mathbb{S} -submodule $\mathcal{R} \subset \mathcal{F}^c(\mathcal{V})^{(2)}$ one can associate a cooperad $\mathcal{Q} = \mathcal{Q}(\mathcal{V}, \mathcal{R})$, the universal suboperad $\mathcal{C} \subset \mathcal{F}^c(\mathcal{V})$ for which the composite

$$\mathcal{C} \hookrightarrow \mathcal{F}^c(\mathcal{V}) \twoheadrightarrow \mathcal{F}^c(\mathcal{V})^{(2)} / \mathcal{R}$$

is zero. The Koszul duality for operads assigns to an operad $\mathcal{P} = \mathcal{P}(\mathcal{V}, \mathcal{R})$ its Koszul dual cooperad $\mathcal{P}^i = \mathcal{Q}(s\mathcal{V}, s^2\mathcal{R})$ and to a cooperad $\mathcal{Q} = \mathcal{Q}(\mathcal{V}, \mathcal{R})$ its Koszul dual operad $\mathcal{Q}^i = \mathcal{P}(s^{-1}\mathcal{V}, s^{-2}\mathcal{R})$. An operad \mathcal{P} is said to be Koszul if the dg operad $\mathcal{F}(s^{-1}\mathcal{P}^i)$ (with the differential encoding the cocomposition map in \mathcal{P}^i) is quasi-isomorphic to \mathcal{P} .

Recall that a structure of a homotopy \mathcal{P} -algebra on a vector space V is exactly the same as a square zero coderivation of the cofree \mathcal{P}^i -coalgebra $\mathcal{P}^i(V)$. The latter coalgebra is referred to as the bar complex of V . A homotopy morphism between two homotopy \mathcal{P} -algebras is the same as a dg \mathcal{P}^i -coalgebra morphism between their bar complexes. The same applies when switching algebras and coalgebras: for a cooperad \mathcal{Q} , a structure of a homotopy \mathcal{Q} -coalgebra on a vector space V is exactly the same as a square zero derivation of the free \mathcal{Q}^i -algebra $\mathcal{Q}^i(V)$. The latter algebra is referred to as the cobar complex of V . A homotopy morphism between two homotopy \mathcal{Q} -coalgebras is the same as a dg \mathcal{Q}^i -algebra morphism between their cobar complexes.

These statements apply to the case when V itself is an operad, that is an algebra over the coloured operad controlling the usual operads; in particular, this translates into the fact that for an \mathbb{S} -module V , a square zero derivation of the free operad $\mathcal{F}(V)$ is the same as a structure of a homotopy cooperad on sV , see [36]. Of course, similarly to how cooperations of an A_∞ -coalgebra are indexed by positive integers (the label of a cooperation describes in how many parts it splits its argument), cooperations of a homotopy cooperad are indexed by trees (the cooperation Δ_t indexed by a tree t describes how to decompose its argument as a cocomposition, decorating internal vertices of t by the parts in that decomposition).

1.2. Homotopy transfer theorem for homotopy cooperads. One of the key features of homotopy structures is that they can be transferred along homotopy retracts. The following result generalising (and dualising) both the homotopy transfer formulae for A_∞ -coalgebras [21, 24] and the homotopy transfer formulae for (pr)operads [16] is proved in [12].

Proposition 1 ([12]). *Let $(\mathcal{C}, \{\Delta_t\})$ be a homotopy cooperad. Let $(\mathcal{H}, d_{\mathcal{H}})$ be a dg \mathbb{S} -module, which is a homotopy retract of the dg \mathbb{S} -module $(\mathcal{C}, d_{\mathcal{C}})$:*

$$H \begin{array}{c} \hookrightarrow \\ \circlearrowleft \end{array} (\mathcal{C}, d_{\mathcal{C}}) \xrightleftharpoons[i]{p} (\mathcal{H}, d_{\mathcal{H}}).$$

The formulae

$$(1) \quad \tilde{\Delta}_t := \sum \pm t(p) \circ ((\Delta_{t_{k+1}} H) \circ_{j_k} (\cdots (\Delta_{t_3} H) \circ_{j_2} ((\Delta_{t_2} H) \circ_{j_1} \Delta_{t_1}))) \circ i,$$

where t is a tree with at least two vertices, and the sum is over all possible ways of writting it by successive insertions of trees with at least two vertices,

$$t = (((t_1 \circ_{j_1} t_2) \circ_{j_2} t_3) \cdots) \circ_{j_k} t_{k+1},$$

and the notation $(\Delta_{t'} H) \circ_j \Delta_t$ means here the composite of Δ_t with $\Delta_{t'} H$ at the j^{th} vertex of the tree t (of course, $\Delta_{t'} H$ is the composite of H and $\Delta_{t'}$), define a homotopy cooperad structure on the dg \mathbb{S} -module $(\mathcal{H}, d_{\mathcal{H}})$ which extends the transferred cocomposition maps $t(p) \circ \Delta_t \circ i$.

1.3. Maurer–Cartan description of homotopy algebras and morphisms. Here we discuss, following [28, 36], a description of homotopy \mathcal{P} -algebras and homotopy morphisms of those algebras in terms of solutions to the Maurer–Cartan equation in a certain L_∞ -algebra.

Let us begin with a general construction of *convolution L_∞ -algebras*. If \mathcal{C} is a homotopy cooperad with the cocomposition map $\Delta_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{F}(\mathcal{C})^{(\geq 2)}$, and \mathcal{P} is a dg operad with the composition map

$\tilde{\mu}_{\mathcal{P}}: \mathcal{F}(\overline{\mathcal{P}})^{(\geq 2)} \rightarrow \mathcal{P}$, the collection $\text{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$ is a homotopy operad, the *convolution homotopy operad* of \mathcal{C} and \mathcal{P} , and hence the direct sum of components of this collection is an L_{∞} -algebra [36]. The structure maps ℓ_n of that L_{∞} -algebra are, for $n > 1$,

$$(2) \quad \ell_n(\phi_1, \dots, \phi_n) = \sum_{\sigma \in \mathbb{S}_n} (-1)^{\text{sgn}(\sigma, \phi_1, \dots, \phi_n)} \tilde{\mu}_{\mathcal{P}} \circ (\phi_{\sigma(1)} \otimes \dots \otimes \phi_{\sigma(n)}) \circ \Delta_n,$$

where Δ_n is the component of $\Delta_{\mathcal{C}}$ which maps \mathcal{C} to $\mathcal{F}(\overline{\mathcal{C}})^{(n)}$, that is the sum of all cooperations Δ_t over trees t with n internal vertices, see [28, 36]. The map ℓ_1 is the usual differential of the space of maps between two chain complexes:

$$\ell_1(\phi) = D(\phi) = d_{\mathcal{P}} \circ \phi - (-1)^{|\phi|} \phi \circ d_{\mathcal{C}}.$$

More generally, if \mathcal{P} is a Koszul operad, A is a homotopy \mathcal{P} -algebra, and B is a dg \mathcal{P}^i -coalgebra, the space $\text{Hom}_{\mathbb{K}}(sB, A)$ acquires a structure of an L_{∞} -algebra, with the structure maps given by similar formulas.

Definition 1. Let \mathfrak{g} be an L_{∞} -algebra with the structure maps ℓ_k , $k \geq 1$. An element $\alpha \in \mathfrak{g}_1$ is said to be a Maurer–Cartan element (notation: $\alpha \in \text{MC}(\mathfrak{g})$) if

$$\sum_{k \geq 1} \frac{1}{k!} \ell_k(\alpha, \alpha, \dots, \alpha) = 0.$$

In the L_{∞} -algebra constructed above, the Maurer–Cartan equation is well defined, since

$$\sum_{k \geq 1} \frac{1}{k!} \ell_k(\alpha, \alpha, \dots, \alpha) = D(\alpha) + \tilde{\mu}_{\mathcal{P}} \circ \mathcal{F}(\alpha) \circ \Delta_{\mathcal{C}}$$

(here, of course, $\mathcal{F}(\alpha): \mathcal{F}(\overline{\mathcal{C}})^{(\geq 2)} \rightarrow \mathcal{F}(\overline{\mathcal{P}})^{(\geq 2)}$ is the map induced by the map $\alpha: \mathcal{C} \rightarrow \mathcal{P}$).

For two graded vector spaces X and Y , the $\{x, y\}$ -coloured endomorphism operad $\text{End}_{X,Y}$ has the components $\text{Hom}_{\mathbb{K}}(X^{\otimes n}, X)$, $\text{Hom}_{\mathbb{K}}(X^{\otimes n}, Y)$, and $\text{Hom}_{\mathbb{K}}(Y^{\otimes n}, Y)$, with the obvious assignment of arities and colours, and equally obvious composition maps. If $\mathcal{P} = \mathcal{F}(\mathcal{V})/(\mathcal{R})$, we define the $\{x, y\}$ -coloured operad $\mathcal{P}_{\bullet \rightarrow \bullet}$ whose algebras are pairs of \mathcal{P} -algebras and a morphism between them as follows. Its generators are two copies of \mathcal{V} , one with all inputs and the output of the colour x , denoted \mathcal{V}_x , and the other with all inputs and the output of the colour y , denoted \mathcal{V}_y , and a unary operation f with the input of the colour x and the output of the colour y . Its relations are \mathcal{R}_x , \mathcal{R}_y , and $f \circ v_x - v_y \circ f^{\otimes n}$ for each $v \in \mathcal{V}(n)$. This operad is homotopy Koszul in the sense of [28]; its minimal model is generated by $s^{-1}\overline{\mathcal{P}}^i_x \oplus s^{-1}\overline{\mathcal{P}}^i_y \oplus \mathcal{P}^i_f$, and the differential can be constructed either via appropriate homotopy transfer formulae [28] or via homological perturbation [27]. The usual warning is in place: though as a collection, the component \mathcal{P}^i_f of the space of generators of the minimal model can be identified with $f \circ \mathcal{P}^i$, the homotopy coloured cooperad structure applied to this subcollection is more complicated than just the naïve splitting of $f \circ \mathcal{P}^i$.

The underlying \mathbb{S} -module of $\overline{\mathcal{P}}^i_x \oplus \overline{\mathcal{P}}^i_y \oplus s\mathcal{P}^i_f$ is a homotopy coloured cooperad, and $\text{End}_{X,Y}$ is a coloured operad, so the general construction of Section 1.3 produces an L_{∞} -algebra structure on the space of \mathbb{S} -module morphisms

$$\text{Hom}_{\mathbb{S}}(\overline{\mathcal{P}}^i_x \oplus \overline{\mathcal{P}}^i_y \oplus s\mathcal{P}^i_f, \text{End}_{X,Y})$$

between them. Clearly, defining such a morphism amounts to defining a triple (h_x, h_y, h_{xy}) in the vector space

$$\text{Hom}_{\mathbb{K}}(\overline{\mathcal{P}}^i(X), X) \oplus \text{Hom}_{\mathbb{K}}(\overline{\mathcal{P}}^i(Y), Y) \oplus \text{Hom}_{\mathbb{K}}(s\mathcal{P}^i(X), Y).$$

The following is proved in [28] for properads, and is essentially present in [21, 36] in the case of operads.

Proposition 2. *A triple of elements (h_x, h_y, h_{xy}) of the vector space*

$$\text{Hom}_{\mathbb{K}}(\overline{\mathcal{P}}^i(X), X) \oplus \text{Hom}_{\mathbb{K}}(\overline{\mathcal{P}}^i(Y), Y) \oplus \text{Hom}_{\mathbb{K}}(s\mathcal{P}^i(X), Y)$$

is a solution to the Maurer–Cartan equation of the L_{∞} -algebra

$$\text{Hom}_{\mathbb{S}}(\overline{\mathcal{P}}^i_x \oplus \overline{\mathcal{P}}^i_y \oplus s\mathcal{P}^i_f, \text{End}_{X,Y})$$

if and only if h_x is a structure of a homotopy \mathcal{P} -algebra on X , h_y is a structure of a homotopy \mathcal{P} -algebra on Y , and $s h_{xy}$ is a homotopy morphism between these algebras.

Moreover, if one fixes the homotopy \mathcal{P} -algebra structures on X and Y , it is possible to describe homotopy morphisms between X and Y in the same language.

Proposition 3. *Suppose that X and Y are two homotopy \mathcal{P} -algebras. There exists a structure of an L_∞ -algebra on*

$$\mathcal{L}(X, Y) := \text{Hom}_{\mathbb{K}}(s\mathcal{P}^i(X), Y)$$

for which solutions to the Maurer–Cartan equation are in one-to-one correspondence with homotopy morphisms between X and Y .

The easiest way to interpret this L_∞ -algebra is, again, as a convolution algebra (of the homotopy \mathcal{P} -algebra Y and the bar complex $\mathcal{P}^i(X)$ viewed as a dg \mathcal{P}^i -coalgebra). The differential on it is given by the formula

$$\ell_1(\phi)(sx) = (d_Y^{(1)} \circ \phi)(sx) + (-1)^{|\phi|}(\phi \circ sD_X)(x),$$

where D_X is the codifferential of the bar complex $\mathcal{P}^i(X)$, and $d_Y^{(1)}$ is the differential of Y . For $k > 1$, the structure maps ℓ_k are given by

$$\ell_k(\phi_1, \dots, \phi_k)(sx) = \sum_{\sigma \in \mathbb{S}_k} \pm (-1)^{\text{sgn}(\sigma, \phi_1, \dots, \phi_k)} (d_Y^{(k)} \circ (\text{id} \otimes \phi_{\sigma(1)} \otimes \dots \otimes \phi_{\sigma(k)}) \circ (1 \otimes s^{\otimes k}) \circ \Delta_X^{k-1})(x),$$

where

$$\Delta_X^{k-1}: \mathcal{P}^i(X) \rightarrow \mathcal{P}^i(k) \otimes_{S_k} \mathcal{P}^i(X)^{\otimes k}$$

is the k^{th} cooperation in the cofree \mathcal{P}^i -coalgebra, and $d_Y^{(k)}: \mathcal{P}^i(k) \otimes_{S_k} Y^{\otimes k} \rightarrow Y$ is the k^{th} corestriction of the codifferential in the bar complex of Y (and where the precise sign \pm is not needed in the following).

2. OVERVIEW OF EXISTING NOTIONS OF HOMOTOPIES

2.1. Concordance. The definition in this section originates from the most classical geometric picture: if $f: X \times I \rightarrow Y$ is a homotopy connecting two manifold maps $p(\cdot) = f(\cdot, 0)$ and $q(\cdot) = f(\cdot, 1)$ between smooth manifolds X and Y , then p and q induce the same map on the cohomology. This is proved by constructing a chain homotopy between p and q . More or less, f induces a morphism of de Rham complexes

$$(3) \quad f^*: \Omega^\bullet(Y) \rightarrow \Omega^\bullet(X) \otimes \Omega^\bullet(I)$$

(if we work in the algebraic category, so that $\Omega^\bullet(X \times I) \simeq \Omega^\bullet(X) \otimes \Omega^\bullet(I)$), and once we denote $f^*(y) = a(x, t) + b(x, t)dt$ and write down the algebra morphism condition, we observe that

$$\dot{a}(x, t) = d_X b + b d_Y,$$

and integrating this equation over I gives

$$q^* - p^* = d_X h + h d_Y,$$

where $h(x) = \int_I b(x, t) dt$. Applying a similar approach to homotopy algebras can go in two different ways (see Equation (3) above, Definition 5.13 in [31], and Definition 2 below): one can either pass to duals of those algebras and work with appropriate cobar complexes instead of de Rham complexes (the advantage being working with algebras rather than with coalgebras) or pass to the dual of the de Rham complex of the interval and work with bar complexes (the advantage being that only one infinite dimensional space needs to be dualised, and less restrictive assumptions on algebras are required). We adopt the latter strategy. Namely, we denote by Ω the de Rham dg algebra of differential forms on the interval $I = [0, 1]$, and by $\Lambda = \Omega^\vee$ the dual dg coalgebra. To make working with infinite dimensional spaces easier, we deal with polynomial differential forms, so that every element of Λ is a (possibly infinite) linear combination of elements $\alpha_i = (t^i)^\vee$, $i \geq 0$ and $\beta_i = (t^i dt)^\vee$, $i \geq 0$. The differential of Λ is defined (in the obvious way) by $d(\alpha_i) = 0$ and $d(\beta_i) = (i+1)\alpha_{i+1}$. The coalgebra structure is defined on the elements α_i, β_i as

$$\begin{aligned} \delta(\alpha_i) &= \sum_{a+b=i} \alpha_a \otimes \alpha_b, \\ \delta(\beta_i) &= \sum_{a+b=i} (\beta_a \otimes \alpha_b + \alpha_a \otimes \beta_b), \end{aligned}$$

and extended to all elements of Λ by linearity (since Ω is infinite dimensional, its dual Λ does not inherit directly a coalgebra structure). The complex Λ is of course quasi-isomorphic to the Čech complex $C_\bullet([0, 1])$ of the unit interval, the chain complex spanned by the elements $\mathbf{0}, \mathbf{1}, \mathbf{01}$ with

$$d(\mathbf{0}) = d(\mathbf{1}) = 0, \quad d(\mathbf{01}) = \mathbf{1} - \mathbf{0}.$$

Moreover, $C_\bullet([0, 1])$ can be realised as a subcomplex of Λ , if one puts

$$(4) \quad \omega(\mathbf{0})(f(t) + g(t) dt) = f(0),$$

$$(5) \quad \omega(\mathbf{1})(f(t) + g(t) dt) = f(1),$$

$$(6) \quad \omega(\mathbf{01})(f(t) + g(t) dt) = \int_0^1 g(t) dt,$$

in other words, $\omega(\mathbf{0}) = \alpha_0$, $\omega(\mathbf{1}) = \sum_{i \geq 0} \alpha_i$, and $\omega(\mathbf{01}) = \sum_{i \geq 0} \frac{\beta_i}{i+1}$.

The following definition (to be more precise, its dual) appears in [31] in the case $\mathcal{P} = \text{Lie}$.

Definition 2 ([31]). Two homotopy morphisms p and q between two homotopy \mathcal{P} -algebras X and Y are said to be *concordant* if there exists a morphism of dg \mathcal{P}^i -coalgebras

$$\phi: \mathcal{P}^i(X) \otimes \Lambda \rightarrow \mathcal{P}^i(Y)$$

for which $p(v) = \phi(v \otimes \omega(\mathbf{0}))$ and $q(v) = \phi(v \otimes \omega(\mathbf{1}))$ whenever $v \in \mathcal{P}^i(X)$. Here the \mathcal{P}^i -coalgebra structure on $\mathcal{P}^i(X) \otimes \Lambda$ comes from the identification $\mathcal{P}^i \simeq \mathcal{P}^i \otimes \text{Com}^c$: since Λ is a Com^c -coalgebra, tensoring with it does not change the type of a coalgebra.

2.2. Homotopy of Maurer–Cartan elements of L_∞ -algebras. In this section, we outline the notions of homotopy between Maurer–Cartan elements of homotopy Lie algebras.

2.2.1. Quillen homotopy. The notion of Quillen homotopy equivalence of Maurer–Cartan elements also uses the de Rham algebra Ω and its two evaluation morphisms $\phi_s: (\Omega, d) \rightarrow (\mathbb{k}, 0)$, $s \in \{0, 1\}$, given by $\phi_s(t) = s$, where t is, as above, the coordinate in I . The motivation for it is geometric: if L is a model of a pointed space Y in the sense of rational homotopy theory, then $L \otimes \Omega$ is a model of the evaluation fibration $ev: \text{map}^*(I, Y) \rightarrow Y$, $ev(\gamma) = \gamma(1)$, as pointed out in [6].

Definition 3. Two Maurer–Cartan elements α_0 and α_1 of an L_∞ -algebra L are said to be *Quillen homotopic* if there exists a Maurer–Cartan element β of the L_∞ -algebra $L \otimes \Omega$ for which $\phi_0(\beta) = \alpha_0$, $\phi_1(\beta) = \alpha_1$.

The following result is well known, and seems to originate in Drinfeld’s letter to Schechtman on deformation theory [11].

Proposition 4. *Let L be an L_∞ -algebra, and A a dg Com-algebra. Then, there is a bijection between the set of Maurer–Cartan elements of the L_∞ -algebra $L \otimes A$ and the set of all dg Com^c -coalgebra morphisms from A^\vee to the bar complex $S^c(sL)$.*

2.2.2. Gauge homotopy. In this section, all infinite series we write make sense under some (local) finiteness or nilpotence conditions. These conditions surely hold for L_∞ -algebras constructed from homotopy cooperads by formulae (2); in general, one should use the language of filtered L_∞ -algebras [29, Sec. 1.3]. The set of Maurer–Cartan elements of an algebra L in that case acquires a structure of a scheme, see [32], which we denote by $\text{MC}(L)$. It is well understood that the right notion of “gauge symmetries” of $\text{MC}(L)$, for L being a dg Lie algebra, is given by the group associated to the Lie algebra L_0 , see [14, 13] for details. So it is natural to look for a similar concept in the general case of L_∞ -algebras.

Proposition 5 ([13, Prop. 4.4]). *Let L be an L_∞ -algebra. Then for each $\alpha \in \text{MC}(L)$, the operations*

$$\ell_k^\alpha(x_1, \dots, x_k) := \sum_{p \geq 0} \frac{1}{p!} \ell_{p+k}(\underbrace{\alpha, \dots, \alpha}_{p \text{ times}}, x_1, \dots, x_k)$$

define a structure of an L_∞ -algebra on the underlying vector space of L .

The following statement is contained in [13]; however, there it is a consequence of much more general results, so for the convenience of the reader we present a more hands-on proof.

Proposition 6. *Let L be an L_∞ -algebra, and $x \in L_0$. The vector field V_x on L_{-1} defined by*

$$V_x(\alpha) = -\ell_1^\alpha(x)$$

is a tangent vector field of the set of Maurer–Cartan elements of L .

Proof. Note that the tangent vectors $\beta \in L_{-1}$ to the set $\text{MC}(L)$ at a point α are characterized by

$$\sum_{p \geq 0} \frac{1}{p!} \ell_{p+1}(\underbrace{\alpha, \dots, \alpha}_p, \beta) = 0,$$

that is

$$\ell_1^\alpha(\beta) = 0$$

(interpret β as an infinitesimal vector). Thus, if β is a tangent vector of $\text{MC}(L)$ at α , we have

$$\ell_1^\alpha(\beta + V_x(\alpha)) = \ell_1^\alpha(\beta - \ell_1^\alpha(x)) = \ell_1^\alpha(\beta) - (\ell_1^\alpha)^2(x) = 0,$$

so that $\beta + V_x(\alpha)$ is tangent as well, which completes the proof. \square

Corollary 1. *If $\alpha \in \text{MC}(L)$, the whole integral curve $\alpha(t)$ of V_x starting at α , that is the solution of the differential equation*

$$\frac{d\alpha}{dt} + \ell_1^\alpha(x) = 0$$

satisfying the initial condition $\alpha(0) = \alpha$, is contained in $\text{MC}(L)$.

This corollary suggests the following definition.

Definition 4. Two Maurer–Cartan elements α_0 and α_1 in an L_∞ -algebra L are said to be *gauge homotopic* if for some $x \in L_0$ there exists an integral curve $\alpha(t)$ of V_x with $\alpha(0) = \alpha_0$ and $\alpha(1) = \alpha_1$.

The integral curve of V_x starting at the given Maurer–Cartan element α_0 can be described by an elegant explicit formula making use of basic combinatorics of rooted trees [13]. However, even without appealing to explicit formulae one can work with this definition of homotopy in an efficient way. For instance, the following result is proved in [25] for dg Lie algebras and in [7] in the full generality for L_∞ -algebras.

Proposition 7 ([7, 25]). *Two Maurer–Cartan elements of an L_∞ -algebra are Quillen homotopic if and only if they are gauge homotopic.*

2.2.3. Cylinder homotopy. The main motivation for the definition of this section is as follows. Consider the quasi-free dg Lie algebra \mathfrak{l} with one generator x of degree -1 and the differential d given by $dx = -\frac{1}{2}[x, x]$. Note that for a dg Lie algebra L the set of Maurer–Cartan elements can be identified with the set of dg Lie algebra morphisms from \mathfrak{l} to L . Thus, if in the homotopy category of dg Lie algebras we can come up with a cylinder object for \mathfrak{l} , the homotopy relation for Maurer–Cartan elements can be defined using that cylinder. It turns out that a right cylinder is given by the *Lawrence–Sullivan construction*.

The Lawrence–Sullivan Lie algebra \mathfrak{L} is a (pronilpotent completion of a) certain quasi-free Lie algebra, that is, a free graded Lie algebra with a differential d of degree -1 satisfying $d^2 = 0$ and the Leibniz rule. It is freely generated by the elements a, b, z , where $|a| = |b| = -1$, $|z| = 0$, and

$$\begin{aligned} da + \frac{1}{2}[a, a] &= db + \frac{1}{2}[b, b] = 0, \\ dz &= [z, b] + \sum_{k \geq 0} \frac{B_k}{k!} \text{ad}_z^k(b - a) = \text{ad}_z(b) + \frac{\text{ad}_z}{\exp(\text{ad}_z) - 1}(b - a), \end{aligned}$$

where the B_k are of course the Bernoulli numbers. It is indeed shown in [6] that this algebra gives the right cylinder object for \mathfrak{l} in the homotopy category of dg Lie algebras, hence the following definition.

Definition 5. Two Maurer–Cartan elements α_0 and α_1 of an L_∞ -algebra L are said to be *cylinder homotopic* if there exists an L_∞ -morphism from \mathfrak{L} to L which takes a to α_0 and b to α_1 .

It turns out that the arising notion of homotopy for Maurer–Cartan elements is equivalent to the other ones available.

Proposition 8 ([5, Prop. 4.5]). *Two Maurer–Cartan elements of an L_∞ -algebra are cylinder homotopic if and only if they are Quillen homotopic.*

2.3. Operadic homotopy. The operadic approach to homotopy algebras [26, 27] is as follows. Recall the $\{x, y\}$ -coloured operad $\mathcal{P}_{\bullet \rightarrow \bullet}$ describing morphisms of \mathcal{P} -algebras, where \mathcal{P} is as usual given by the quadratic presentation $\mathcal{P} = \mathcal{F}(\mathcal{V})/(\mathcal{R})$. It has as its generators two copies of \mathcal{V} which we shall denote \mathcal{V}_x and \mathcal{V}_y , and a unary operation f . Its relations are $\mathcal{R}_x, \mathcal{R}_y$, and $f \circ v_x - v_y \circ f^{\otimes n}$ for each $v \in \mathcal{V}(n), n \geq 1$. Using homological perturbation, Markl [27] proves the following results. By the leading terms in an expansion of some term λ both theorems mean terms which only involve unary generating operations and the generating operations of the same arity as λ .

Proposition 9 ([27, Th. 7]). *The minimal model of the $\{x, y\}$ -coloured operad $\mathcal{P}_{\bullet \rightarrow \bullet}$ has the space of generators*

$$s^{-1}\overline{\mathcal{P}}_x \oplus s^{-1}\overline{\mathcal{P}}_y \oplus \overline{\mathcal{P}}_f \oplus f,$$

and the differential ∂ satisfies $\partial(f) = 0$ and has the “leading terms”

$$\partial(M_f) = f \circ (s^{-1}M_x) - (s^{-1}M_y) \circ (f \otimes \cdots \otimes f) + \cdots.$$

Proposition 10 ([27, Th. 18]). *There exists a quasi-free resolution of the $\{x, y\}$ -coloured operad $\mathcal{P}_{\bullet \rightarrow \bullet}$ whose space of generators is*

$$s^{-1}\overline{\mathcal{P}}_x \oplus s^{-1}\overline{\mathcal{P}}_y \oplus \overline{\mathcal{P}}_p \oplus \overline{\mathcal{P}}_q \oplus s\overline{\mathcal{P}}_h \oplus p \oplus q \oplus h,$$

where $|h| = 1$, and the differential ∂ satisfies $\partial(p) = \partial(q) = 0$, $\partial(h) = p - q$ and has the “leading terms”

$$(7) \quad \partial(M_p) = p \circ (s^{-1}M_x) - (s^{-1}M_y) \circ (p \otimes \cdots \otimes p) + \cdots,$$

$$(8) \quad \partial(M_q) = q \circ (s^{-1}M_x) - (s^{-1}M_y) \circ (q \otimes \cdots \otimes q) + \cdots,$$

$$(9) \quad \partial(sM_h) = M_p - M_q - h \circ (s^{-1}M_x) + (-1)^{\deg(M)-1} (s^{-1}M_y) \circ [[h]] + \cdots,$$

where

$$[[h]] = \text{Sym} \left(h \otimes q^{\otimes(n-1)} + p \otimes h \otimes q^{\otimes(n-2)} + \cdots + p^{\otimes(n-1)} \otimes h \right).$$

In the preceding propositions the non-leading terms ‘ \cdots ’ are not explicitly given. Hence, these propositions do not immediately lead to a notion of homotopy morphism, or homotopy between homotopy morphisms. Nevertheless, the first of them matches the notion of a homotopy morphism between two homotopy \mathcal{P} -algebras as a morphism of dg \mathcal{P}^i -coalgebras between their bar complexes, in the sense that the operadic reformulation of the latter yields a minimal model with a differential of the required shape. Of course, this story starts even at an earlier stage, since the minimal model of \mathcal{P} is built on the set of generators $s^{-1}\overline{\mathcal{P}}^i$ (since \mathcal{P} is assumed Koszul). As concerns the second proposition, only the existence theorem for such a resolution is proved in [27], and it is obvious that such a model cannot be unique in any reasonable sense (it is not minimal by the construction, and hence there is lots of freedom when reconstructing the lower terms ‘ \cdots ’ for the differential). In one example, the nonsymmetric operad of associative algebras, explicit formulae for images of the generators under the differential were computed in [27], and it was observed that this leads to the notion of derivation homotopy, as in [17]. Below, we shall explain how to obtain an explicit resolution of the prescribed type in a natural way, thus providing an honest notion of operadic homotopy.

Remark 1. The formula for $[[h]]$ makes one think of derivation homotopies as well, but this intuition is only correct under very restrictive assumptions, e.g., for the case $\mathcal{P} = \text{Lie}$ the derivation homotopy formulae only work if the cobar complex of the target algebra is free as a differential algebra (i.e., the structure maps ℓ_k of the target algebra vanish for $k > 1$), see [31, 34]. Nonetheless the derivation homotopy formulae do always work for nonsymmetric operads. The reason for that is that the Čech complex carries a structure of a dg coassociative coalgebra given by

$$\Delta(0) = 0 \otimes 0, \quad \Delta(1) = 1 \otimes 1,$$

$$\Delta(01) = 0 \otimes 01 + 01 \otimes 1,$$

and so can be used in place of Λ whenever our operad is nonsymmetric, and tensoring with the coassociative coalgebra does not change the type of algebras. These formulae for the coproduct of course naturally lead to derivation homotopies.

3. CONCORDANCE AND QUILLEN HOMOTOPY

Our main observation allowing to relate the notion of concordance to the notion of a Quillen homotopy is as follows.

Theorem 1. *Let X and Y be two homotopy \mathcal{P} -algebras. For every dg Com^c -coalgebra A , we have*

$$(10) \quad \text{Hom}_{dg-\text{Com}^c-\text{coalg}}(A, \text{Com}^c(s\mathcal{L}(X, Y))) \simeq \text{Hom}_{dg-\mathcal{P}^i-\text{coalg}}(\mathcal{P}^i(X) \otimes A, \mathcal{P}^i(Y)),$$

where $\mathcal{P}^i(X)$ and $\mathcal{P}^i(Y)$ are, as dg \mathcal{P}^i -coalgebras, the bar complexes of X and Y respectively.

Proof. Since a morphism from any cocommutative coalgebra to the free cocommutative coalgebra is uniquely determined by its corestriction on cogenerators, we have

$$\text{Hom}_{dg-\text{Com}^c-\text{coalg}}(A, \text{Com}^c(s\mathcal{L}(X, Y))) \subset \text{Hom}_{\mathbb{k}}(A, s\mathcal{L}(X, Y)),$$

and similarly

$$\text{Hom}_{dg-\mathcal{P}^i-\text{coalg}}(\mathcal{P}^i(X) \otimes A, \mathcal{P}^i(Y)) \subset \text{Hom}_{\mathbb{k}}(\mathcal{P}^i(X) \otimes A, Y).$$

Also, because of the description of our L_∞ -algebra in Proposition 3, we have

$$\text{Hom}_{\mathbb{k}}(A, s\mathcal{L}(X, Y)) = \text{Hom}_{\mathbb{k}}(A, \text{Hom}_{\mathbb{k}}(\mathcal{P}^i(X), Y)).$$

Notice that there exists a natural isomorphism

$$\text{Hom}_{\mathbb{k}}(A, \text{Hom}_{\mathbb{k}}(\mathcal{P}^i(X), Y)) \simeq \text{Hom}_{\mathbb{k}}(\mathcal{P}^i(X) \otimes A, Y)$$

given, for $\alpha \in \text{Hom}_{\mathbb{k}}(A, \text{Hom}_{\mathbb{k}}(\mathcal{P}^i(X), Y))$, $a \in A$, $p \in \mathcal{P}^i(X)$ by the formula

$$\hat{\alpha}(p \otimes a) := (-1)^{|p||a|} \alpha(a)(p).$$

Therefore the spaces we want to identify are embedded in the same vector space. It remains to check that the actual equations that define these spaces, that is compatibility with differentials, actually match. If we take $\alpha \in \text{Hom}_{\mathbb{k}}(A, s\mathcal{L}(X, Y))$, then the compatibility with the differentials means that

$$(\alpha \circ d_A)(a) = \sum_k (d_{\mathcal{L}(X, Y)}^{(k)} \circ S^k(\alpha) \circ \Delta_A^{k-1})(a)$$

for all $a \in A$. Here $\Delta_A: A \rightarrow S^2(A)$ is the coproduct of A , and $d_{\mathcal{L}(X, Y)}^{(k)}: \text{Com}^c(s\mathcal{L}(X, Y)) \rightarrow s\mathcal{L}(X, Y)$ is the k^{th} corestriction of the codifferential of the bar complex. Recalling the explicit formulae for the L_∞ -algebra structure on $\mathcal{L}(X, Y)$, we can rewrite the preceding equation as

$$(11) \quad (\alpha \circ d_A)(a) = -(-1)^{|a|} \alpha(a) \circ D_X + \sum_k d_Y^{(k)} \circ (\text{id} \otimes (S^k(\alpha) \circ \Delta_A^{k-1}(a))) \circ \Delta_X^{k-1}.$$

For $\beta \in \text{Hom}_{\mathbb{k}}(\mathcal{P}^i(X) \otimes A, Y)$, the compatibility with differentials means that

$$\sum_k (d_Y^{(k)} \circ (\text{id} \otimes \beta^{\otimes k}) \circ \iota \circ (\Delta_X^{k-1} \otimes \Delta_A^{k-1}))(p \otimes a) = \beta(D_X(p) \otimes a + (-1)^{|p|} p \otimes d_A(a)),$$

for all $a \in A$, $p \in \mathcal{P}^i(X)$. Here $d_Y^{(k)}$ is the k^{th} corestriction of the differential of the bar complex of Y , and D_X is the differential of the bar complex of X , and ι is the embedding

$$(\mathcal{P}^i(k) \otimes_{S_k} \mathcal{P}^i(X)^{\otimes k}) \otimes S^k(A) \hookrightarrow \mathcal{P}^i(k) \otimes_{S_k} (\mathcal{P}^i(X) \otimes A)^{\otimes k}$$

used to define the \mathcal{P}^i -coalgebra structure on $\mathcal{P}^i(X) \otimes A$. It remains to note that if $\beta = \hat{\alpha}$, we obtain

$$\sum_k (d_Y^{(k)} \circ (\text{id} \otimes \hat{\alpha}^{\otimes k}) \circ \iota \circ (\Delta_X^{k-1} \otimes \Delta_A^{k-1}))(p \otimes a) = \hat{\alpha}(D_X(p) \otimes a + (-1)^{|p|} p \otimes d_A(a)),$$

that is

$$\begin{aligned} (-1)^{|p||a|} \sum_k (d_Y^{(k)} \circ (\text{id} \otimes (S^k(\alpha) \circ \Delta_A^{k-1}(a))) (\Delta_X^{k-1}(p))) = \\ = (-1)^{(|p|+1)|a|} \alpha(a)(D_X(p)) + (-1)^{|p|+|p|(|a|+1)} \alpha(d_A(a))(p), \end{aligned}$$

which is immediately identified with Condition (11), and the theorem follows. \square

Corollary 2. *Two homotopy morphisms of homotopy \mathcal{P} -algebras X and Y are concordant if and only if the corresponding Maurer–Cartan elements of $\mathcal{L}(X, Y)$ are Quillen homotopic.*

Proof. By Proposition 4, Maurer–Cartan elements of the L_∞ -algebra $\mathcal{L}(X, Y) \otimes \Omega$ are in one-to-one correspondence with the dg Com^c -coalgebra morphisms between Λ and $\text{Com}^c(s\mathcal{L}(X, Y))$. The latter, as we just proved, are in one-to-one correspondence with dg \mathcal{P}^i -coalgebra morphisms from $\mathcal{P}^i(X) \otimes \Lambda$ to $\mathcal{P}^i(Y)$, and we are done. \square

4. CONCORDANCE AND OPERADIC HOMOTOPY

To relate the notion of concordance to the operadic notion of a homotopy, we shall begin with expressing the former operadically. A morphism of \mathcal{P}^i -coalgebras as above is obviously completely defined by its corestrictions

$$\mathcal{P}^i(X) \otimes \Lambda \rightarrow Y.$$

Thus, the datum of two homotopy \mathcal{P} -algebras, two homotopy morphisms between them, and a homotopy between those morphisms can be defined operadically as follows.

Theorem 2. *The datum of two homotopy \mathcal{P} -algebras and two concordant homotopy morphisms between them can be encoded by the structure of an algebra over a quasi-free $\{x, y\}$ -coloured operad*

$$\mathcal{P}_{\bullet \rightarrow \bullet, \Omega} := (\mathcal{F}(\mathcal{W}), d)$$

whose space of generators is

$$\mathcal{W} = s^{-1}\overline{\mathcal{P}}^i_x \oplus s^{-1}\overline{\mathcal{P}}^i_y \oplus \mathcal{P}^i_{x \rightarrow y} \otimes \Lambda.$$

Here the subscript x denotes the operations whose all inputs as well as the output have the colour x (corresponding to the first homotopy \mathcal{P} -algebra), the colour y — the operations whose all inputs as well as the output have the colour y (corresponding to the second homotopy \mathcal{P} -algebra), and the subscript $x \rightarrow y$ — the operations whose all inputs have the colour x and the output has the colour y .

Proof. The statement is fairly obvious. Indeed, these generators merely come from the fact that $s^{-1}\overline{\mathcal{P}}^i_x$ and $s^{-1}\overline{\mathcal{P}}^i_y$ describe the corestrictions of the coderivations on the bar complexes of our algebras, and $\mathcal{P}^i_{x \rightarrow y} \otimes \Lambda$ describes the corestrictions of a ‘family’ of coalgebra morphisms. The only constraints are that the coderivations be codifferentials and that the coalgebra morphisms be compatible with the coderivations. This gives rise to a differential on the corresponding free operad. \square

Remark 2. As we discussed above, a differential of the free operad $\mathcal{F}(\mathcal{W})$ is the same as a structure of a cooperad up to homotopy on $s\mathcal{W}$. For our purposes, this description will be more important, so we give it here, leaving it as an exercise to the reader to check that this definition comes from unraveling the codifferential and dg-coalgebra morphism conditions. Namely, we have

$$s\mathcal{W} = \overline{\mathcal{P}}^i_x \oplus \overline{\mathcal{P}}^i_y \oplus s\mathcal{P}^i_{x \rightarrow y} \otimes \Lambda,$$

and the cocompositions Δ_t , for each tree t , are already defined on the summands $\overline{\mathcal{P}}^i_x$ and $\overline{\mathcal{P}}^i_y$, those being cooperads by construction (since those are honest cooperads, not cooperads up to homotopy, most of the cocompositions will vanish). It remains to define the cocompositions on $s\mathcal{P}^i_{x \rightarrow y} \otimes \Lambda$. Those are made of two ingredients, cocompositions on $s\mathcal{P}^i_{x \rightarrow y}$, and decorations of those by tensor products of elements of Λ arising as tensor factors in iterated coproducts on Λ . The former cocompositions that do not vanish are of two types only: splitting an element $sF \in s\mathcal{P}^i_{x \rightarrow y}$ as $sH \circ_i G \in s\mathcal{P}^i_{x \rightarrow y} \circ_i \overline{\mathcal{P}}^i_x$, and splitting an element $sF \in s\mathcal{P}^i_{x \rightarrow y}$ as $G(sH_1, \dots, sH_p) \in \overline{\mathcal{P}}^i_y \circ s\mathcal{P}^i_{x \rightarrow y}$, coming with appropriate signs.

An important fact [8, 5, 6] which is a key ingredient in our approach is as follows.

Proposition 11. *$(C_\bullet([0, 1]), d)$ is a homotopy retract of (Λ, d) .*

We present a proof here both because its details are crucial for our main computation. Also, some signs in formulae are different from those in [5, 6] since our motivation comes from the ‘geometric’ wish of identifying a subcomplex of chains of the unit interval inside Λ . In fact, the formulae for the contraction K below are precisely the duals of those for the appropriate Dupont’s contraction [10, 8, 13].

Proof. To interpret $C_\bullet([0, 1])$ as a homotopy retract of Λ we exhibit a diagram

$$\kappa \begin{array}{c} \curvearrowright \\ \end{array} (\Lambda, d) \begin{array}{c} \xleftarrow{\theta} \\ \xrightarrow{\omega} \end{array} (C_\bullet([0, 1]), d),$$

with $\theta\omega = \text{id}_{C_\bullet([0,1])}$, $\text{id}_\Lambda - \omega\theta = dK + Kd$, $\theta K = K\omega = K^2 = 0$. For that, we put

$$\theta(\alpha_i) = \begin{cases} \mathbf{0}, & i = 0, \\ \mathbf{1} - \mathbf{0}, & i = 1, \\ 0, & i > 1, \end{cases}$$

and

$$\theta(\beta_i) = \begin{cases} \mathbf{0}\mathbf{1}, & i = 0, \\ 0, & i > 0, \end{cases}$$

so that for the inclusion ω defined by Formulae (4)–(6) above we have

$$\begin{aligned} \theta\omega(\mathbf{0}) &= \theta(\alpha_0) = \mathbf{0}, \\ \theta\omega(\mathbf{1}) &= \theta\left(\sum_{i \geq 0} \alpha_i\right) = \mathbf{0} + \mathbf{1} - \mathbf{0} = \mathbf{1}, \\ \theta\omega(\mathbf{0}\mathbf{1}) &= \theta\left(\sum_{i \geq 0} \frac{\beta_i}{i+1}\right) = \mathbf{0}\mathbf{1}, \end{aligned}$$

yielding $\theta\omega = \text{id}_{C_\bullet([0,1])}$. We also define a map $K: \Lambda \rightarrow \Lambda$ by putting $K(\beta_i) = 0$ and

$$(12) \quad K(\alpha_i) = \begin{cases} 0, & i = 0, \\ -\sum_{j \geq 1} \frac{\beta_j}{j+1}, & i = 1, \\ \frac{\beta_{i-1}}{i}, & i > 1. \end{cases}$$

Then $\theta K = K\omega = K^2 = 0$, and

$$(dK + Kd)(\alpha_i) = dK(\alpha_i) = \begin{cases} 0, & i = 0, \\ -\sum_{j \geq 2} \alpha_j, & i = 1, \\ \alpha_i, & i > 1, \end{cases}$$

and

$$(dK + Kd)(\beta_i) = Kd(\beta_i) = K((i+1)\alpha_{i+1}) = \begin{cases} -\sum_{j \geq 1} \frac{\beta_j}{j+1}, & i = 0, \\ \beta_i, & i > 0. \end{cases}$$

It remains to notice that

$$(\text{id}_\Lambda - \omega\theta)(\alpha_i) = \begin{cases} 0, & i = 0, \\ -\sum_{i \geq 2} \alpha_i, & i = 1, \\ \alpha_i, & i > 1, \end{cases}$$

and

$$(\text{id}_\Lambda - \omega\theta)(\beta_i) = \begin{cases} -\sum_{j \geq 1} \frac{\beta_j}{j+1}, & i = 0, \\ \beta_i, & i > 0, \end{cases}$$

so $dK + Kd = \text{id}_\Lambda - \omega\theta$, as required. \square

We can use the preceding homotopy retract to transfer various structures from Λ . As a toy model, let us recall how one can recover (the universal enveloping algebra of) the Lawrence-Sullivan dg Lie algebra \mathfrak{L} using this retract. The enveloping algebra of \mathfrak{L} is the quasi-free dg associative algebra with generators a, b, z , where $|a| = |b| = -1$, $|z| = 0$. This data is equivalent to the structure of an A_∞ -coalgebra on the suspension of the space of generators, that is the space spanned by the elements $u = sa$, $v = sb$, $w = sz$,

where $|u| = |v| = 0$, $|w| = 1$. Explicitly this A_∞ -coalgebra is given by

$$(13) \quad \delta_1(w) = u - v, \delta_1(u) = \delta_1(v) = 0,$$

$$(14) \quad \delta_2(w) = -\frac{1}{2}w \otimes (u + v) - \frac{1}{2}(u + v) \otimes w, \delta_2(u) = -u \otimes u, \delta_2(v) = -v \otimes v,$$

$$(15) \quad \delta_k(w) = - \sum_{p+q=k-1} \frac{b_{k-1}}{p!q!} w^{\otimes p} \otimes (u - v) \otimes w^{\otimes q}, \quad \delta_k(u) = \delta_k(v) = 0, k \geq 3.$$

The following result is proved in [5]; in fact, it is the dual version of the statement proved earlier in [8] for the homotopy transfer of C_∞ -structures between (Ω, d) and its subcomplex spanned by $1, t, dt$.

Proposition 12 ([8, 5, 6]). *The C_∞ -coalgebra structure on $\langle u, v, w \rangle \simeq C_\bullet([0, 1])$ given by (13)–(15) is obtained from the dg coalgebra structure on Λ by homotopy transfer formulae.*

We now formulate our main theorem, which we also prove by means of a transfer based upon the mentioned homotopy retract. The quasi-free operad $(\mathcal{F}(\mathcal{W}), d)$ from Theorem 2 gives rise to a structure of a homotopy cooperad on the dg \mathbb{S} -module

$$s\mathcal{W} = \overline{\mathcal{P}}_x^i \oplus \overline{\mathcal{P}}_y^i \oplus s\mathcal{P}_{x \rightarrow y}^i \otimes \Lambda.$$

Theorem 3. *The dg \mathbb{S} -module $s\mathcal{W}$ admits*

$$s\mathcal{W}_0 := \overline{\mathcal{P}}_x^i \oplus \overline{\mathcal{P}}_y^i \oplus s\mathcal{P}_{x \rightarrow y}^i \otimes C_\bullet([0, 1])$$

as a homotopy retract, and the induced homotopy cooperad structure on $s\mathcal{W}_0$ is of the type described in Proposition 10.

Proof. Note that the differential of the chain complex

$$s\mathcal{W} = \overline{\mathcal{P}}_x^i \oplus \overline{\mathcal{P}}_y^i \oplus s\mathcal{P}_{x \rightarrow y}^i \otimes \Lambda$$

comes precisely from the dual of the de Rham differential on $\Lambda = \Omega^\vee$. Thus, the homotopy retract

$$K \begin{array}{c} \curvearrowright \\ \left(\Lambda, d \right) \xrightleftharpoons[\omega]{\theta} \left(C_\bullet([0, 1]), d \right) \end{array},$$

constructed in Proposition 11 gives rise to a homotopy retract

$$H \begin{array}{c} \curvearrowright \\ \left(\overline{\mathcal{P}}_x^i \oplus \overline{\mathcal{P}}_y^i \oplus s\mathcal{P}_{x \rightarrow y}^i \otimes \Lambda, d \right) \xrightleftharpoons[i]{p} \left(\overline{\mathcal{P}}_x^i \oplus \overline{\mathcal{P}}_y^i \oplus s\mathcal{P}_{x \rightarrow y}^i \otimes C_\bullet([0, 1]), d \right) \end{array},$$

where

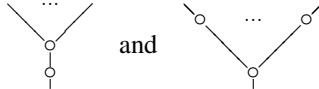
$$H(v_1, v_2, sv_3 \otimes \lambda) = (0, 0, sv_3 \otimes K(\lambda)),$$

$$i(v_1, v_2, sv_3 \otimes c) = (v_1, v_2, sv_3 \otimes \omega(c)),$$

$$p(v_1, v_2, sv_3 \otimes \lambda) = (v_1, v_2, sv_3 \otimes \theta(\lambda)).$$

It remains to apply Formulae (1) to compute the transferred operations.

Note that the leading terms in Formulae (7)–(9), apart from the indecomposable terms which clearly correspond to the differential $d(\mathbf{01}) = \mathbf{1} - \mathbf{0}$ of the Čech complex, come from the cooperations indexed by trees t



which cannot be represented as nontrivial substitutions of trees with at least two vertices, so all the homotopy transfer computations simplify drastically, and the transferred cooperad maps $\tilde{\Delta}_t$ are given by the naive formula $\tilde{\Delta}_t = t(p) \circ \Delta_t \circ i$. Let us show how Formulae (7)–(9) arise naturally.

To recover Formula (7), we note that

$$\delta^{n-1}(\omega(\mathbf{0})) = \delta^{n-1}(\alpha_0) = \alpha_0^{\otimes n} = \omega(\mathbf{0})^{\otimes n},$$

so when computing the homotopy cooperad cocompositions $\tilde{\Delta}_t$ on

$$M \otimes \mathbf{0} \in \mathcal{P}_{x \rightarrow y}^i \otimes C_\bullet([0, 1]),$$

the only contributions to the leading terms are

$$\mathbf{0} \circ_1 M \in (\mathcal{P}_{x \rightarrow y}^i(1) \otimes C_\bullet([0, 1])) \circ_1 \mathcal{P}_x^i$$

and

$$M \circ \mathbf{0}^{\otimes n} \in \mathcal{P}_y^i \circ (\mathcal{P}_{x \rightarrow y}^i(1) \otimes C_\bullet([0, 1])).$$

To recover Formula (8), we note that

$$\delta(\omega(\mathbf{1})) = \delta \left(\sum_{i \geq 0} \alpha_i \right) = \sum_{i \geq 0} \sum_{a+b=i} \alpha_a \otimes \alpha_b = \omega(\mathbf{1}) \otimes \omega(\mathbf{1}),$$

so $\delta^{n-1}(\omega(\mathbf{1})) = \omega(\mathbf{1})^{\otimes n}$, and hence when computing the homotopy cooperad cocompositions $\tilde{\Delta}_t$ on $M \otimes \mathbf{1} \in \mathcal{P}_{x \rightarrow y}^i \otimes C_\bullet([0, 1])$, the only contributions to the leading terms are

$$\mathbf{1} \circ_1 M \in (\mathcal{P}_{x \rightarrow y}^i(1) \otimes C_\bullet([0, 1])) \circ_1 \mathcal{P}_x^i$$

and

$$M \circ \mathbf{1}^{\otimes n} \in \mathcal{P}_y^i \circ (\mathcal{P}_{x \rightarrow y}^i(1) \otimes C_\bullet([0, 1])).$$

To recover Formula (9), a bit more work is required. We wish to investigate the transferred homotopy cooperad cocompositions $\tilde{\Delta}_t$ evaluated on elements

$$M \otimes \mathbf{01} \in \mathcal{P}_{x \rightarrow y}^i \otimes C_\bullet([0, 1]).$$

Of course, we instantly recover the leading term

$$\mathbf{01} \circ_1 M \in (\mathcal{P}_{x \rightarrow y}^i(1) \otimes C_\bullet([0, 1])) \circ_1 \mathcal{P}_x^i,$$

since in that case no nontrivial computations within the coalgebra Λ occur. However, for the leading term that lands in $\mathcal{P}_y^i \circ (\mathcal{P}_{x \rightarrow y}^i(1) \otimes C_\bullet([0, 1]))$, the computation is less obvious. Let us assume that $M \in \mathcal{P}_{x \rightarrow y}^i(n)$. The $C_\bullet([0, 1])$ -decoration of the corresponding leading term is precisely

$$(\theta^{\otimes n} \circ \delta^{n-1} \circ \omega)(\mathbf{01}).$$

Let us compute that decoration explicitly. We have

$$\begin{aligned} (\theta^{\otimes n} \circ \delta^{n-1} \circ \omega)(\mathbf{01}) &= (\theta^{\otimes n} \circ \delta^{n-1}) \left(\sum_{i \geq 0} \frac{\beta_i}{i+1} \right) = \\ &= \sum_{i \geq 0} \frac{1}{i+1} \theta^{\otimes n} \left(\sum_{i_1 + \dots + i_n = i} \sum_{j=1}^n \alpha_{i_1} \otimes \dots \otimes \alpha_{i_{j-1}} \otimes \beta_{i_j} \otimes \alpha_{i_{j+1}} \otimes \dots \otimes \alpha_{i_n} \right). \end{aligned}$$

Let us concentrate on the term $j = n$ in the third sum for the moment. Recalling the definition of θ , we conclude that we must have $i_n = 0$, and $i_k \in \{0, 1\}$ for $k < n$. Together with the condition $i_1 + \dots + i_n = i$, this means that after applying θ we end up with a sum over all i -element subsets of $\{1, \dots, n-1\}$, and the tensor product has $\mathbf{1} - \mathbf{0}$ on the places indexed by the given subset, and $\mathbf{0}$ otherwise. Since the total sum obviously lands in the subspace of tensors symmetric in the first $n-1$ factors, we may rewrite it as

$$\begin{aligned} \sum_{i \geq 0} \frac{1}{i+1} \binom{n-1}{i} \mathbf{0}^{\odot n-1-i} \odot (\mathbf{1} - \mathbf{0})^{\odot i} \otimes \mathbf{01} &= \\ &= \sum_{i \geq 0} \frac{1}{n} \binom{n}{i+1} \mathbf{0}^{\odot n-1-i} \odot (\mathbf{1} - \mathbf{0})^{\odot i} \otimes \mathbf{01} = \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{0}^{\odot n-1-j} \odot \mathbf{1}^{\odot j} \otimes \mathbf{01}. \end{aligned}$$

Here we used the formulae $\frac{1}{i+1} \binom{n-1}{i} = \frac{1}{n} \binom{n}{i+1}$ and

$$\sum_{i=0}^{n-1} \binom{n}{i+1} a^{n-1-i} b^i = \sum_{j=0}^{n-1} a^{n-1-j} (a+b)^j,$$

the latter valid in any commutative ring (and is proved in $\mathbb{Z}[a, b]$ by noticing that both the left hand side and the right hand side are equal to the same expression $\frac{(a+b)^n - a^n}{b}$).

Now we recall the contributions of all individual $j = 1, \dots, n$ above, and notice that the factor $\frac{1}{n}$ precisely contributes to creating from all these contributions the term

$$\sum_{j=0}^{n-1} \mathbf{0}^{\odot n-1-j} \odot \mathbf{01} \odot \mathbf{1}^{\odot j}.$$

This is exactly the same as the element

$$[[h]] = \text{Sym} \left(h \otimes q^{\otimes(n-1)} + p \otimes h \otimes q^{\otimes(n-2)} + \dots + p^{\otimes(n-1)} \otimes h \right)$$

appearing in Formula (9), which completes the proof. The analysis of signs is left to the reader. (In fact, instead of analysing the signs in transfer formulae, one can note that there is exactly one choice of signs for which the formulae (7)–(9) could possibly work with the prescribed types of leading terms, so there is nothing to prove.) \square

Remark 3. Since Equation (12) implies that $K(\alpha_0) = K(\sum_{i \geq 0} \alpha_i) = 0$, it is easy to see that we not only recover Formulae (7) and (8), but in fact see that the homotopy transfer formulae for the cooperations $\tilde{\Delta}_t$ evaluated on elements

$$M \otimes \mathbf{0}, M \otimes \mathbf{1} \in \mathcal{P}_{x \rightarrow y}^i \otimes C_\bullet([0, 1])$$

are precisely what we expect, that is duals of the dg \mathcal{P}^i -morphism conditions for the corresponding maps of bar complexes. For the elements

$$M \otimes \mathbf{01} \in \mathcal{P}_{x \rightarrow y}^i \otimes C_\bullet([0, 1]),$$

the formulae obtained by homotopy transfer are more complicated, but however very explicit, as a combination of the fact that before the transfer the homotopy cooperad structure was somewhat degenerate and the fact that the maps p and H vanish on many elements involved.

We denote by $\mathcal{P}_{\bullet \rightarrow \bullet, \infty}$ the quasi-free operad $(\mathcal{F}(\mathcal{W}_0), d)$ encoding the homotopy cooperad structure on $s\mathcal{W}_0$ computed in the proof of Theorem 3. This theorem implies that the operad $\mathcal{P}_{\bullet \rightarrow \bullet, \infty}$ is a good candidate to encode the operadic notion of homotopy between homotopy morphisms of homotopy \mathcal{P} -algebras.

Corollary 3. *The notion of operadic homotopy is homotopically equivalent to the notion of concordance. More precisely, we have the equivalence of homotopy categories of algebras*

$$\text{Ho}(\mathcal{P}_{\bullet \rightarrow \bullet, \infty}) \cong \text{Ho}(\mathcal{P}_{\bullet \rightarrow \bullet, \Omega}).$$

Proof. Since the operads $\mathcal{P}_{\bullet \rightarrow \bullet, \Omega}$ and $\mathcal{P}_{\bullet \rightarrow \bullet, \infty}$ are cofibrant and split (because we work in characteristic zero), Theorem 4.7.4 of [18] applies. \square

5. FURTHER DIRECTIONS

One possible direction where our homotopy transfer approach might be useful for “de-mystifying” the story is a conjecture made in the end of [27]. That conjecture suggests, for every operad \mathcal{P} admitting a minimal model $(\mathcal{F}(\mathcal{V}), d)$ and every small category \mathcal{C} with a chosen cofibrant replacement $(F(\mathcal{W}), \partial)$ of $\mathbb{k}\mathcal{C}$, the existence of a cofibrant replacement

$$(\mathcal{F}(\mathcal{V} \otimes \mathbb{k} \text{Ob}(\mathcal{C}) \oplus \mathcal{W} \oplus s\mathcal{V} \otimes \mathcal{W}), d)$$

for any coloured operad $\mathcal{O}_{\mathcal{P}, \mathcal{Q}}$ describing \mathcal{P} -algebras and morphisms between them that form a diagram of shape \mathcal{Q} . The differential d of this replacement is conjectured to have a specific shape [27]. A somewhat restricted version of this conjecture is proved in the case of a Koszul operad \mathcal{P} in [9]. We hope that homotopy transfer techniques might be the right tool to prove this conjecture in full generality in the Koszul case.

Another natural question to address in future work is to apply homotopy transfer theorems for homotopy retracts from de Rham complexes to Čech complexes beyond the case of the interval. It would be interesting already in the case of contractible spaces, for example for higher-dimensional simplexes and

higher-dimensional disks the corresponding computation would contain further information on the higher dimensional categorification of algebras.

Further, while we concentrated on the case of a Koszul operad \mathcal{P} , it would be interesting to generalise the relevant notions and result to the case of any operad admitting a minimal cofibrant replacement $(\mathcal{F}(\mathcal{V}), d)$, putting $\mathcal{P}^i := s\mathcal{V}$, and making appropriate adjustments in the view of the fact that \mathcal{P}^i is no longer an honest cooperad but rather a homotopy cooperad.

Finally, using the results of the present paper, we are investigating, in works in progress, homotopies of homotopy morphisms of homotopy Loday algebras [1], homotopies of morphisms of Lie n -algebroids [3] and of Loday algebroids [15].

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